Genus two surface and quarter BPS dyons: the contour prescription

JHEP03(2009)151

(http://iopscience.iop.org/1126-6708/2009/03/151)

Published by IOP Publishing for SISSA

Received: *February 6, 2009*
Accepted: *March 1, 2009*
Published: *March 31, 2009*

JHEP03(2009)151

Genus two surface and quarter BPS dyons: the contour prescription

Shamik Banerjee, Ashoke Sen and Yogesh K. Srivastava

Harish-Chandra Research Institute,
Chhatnag Road, Jhusi, Allahabad 211019, India

E-mail: bshamik@mri.ernet.in, sen@mri.ernet.in, yogesh@mri.ernet.in

Abstract: Following the suggestion of hep-th/0506249 and hep-th/0612011, we represent quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories as string network configuration and explore the role of genus two surfaces in determining the spectrum of such dyons. Our analysis leads to the correct contour prescription for integrating the partition function to determine the spectrum in different domains of the moduli space separated by the walls of marginal stability.

Keywords: Black Holes in String Theory, p-branes, D-branes, M-Theory

ArXiv ePrint: 0808.1746

doi:10.1088/1126-6708/2009/03/151

Contents

1 Introduction

We now have a good understanding of the spectrum of quarter BPS dyons in a variety of $\mathcal{N} = 4$ supersymmetric string theories [1–28]. One of the mysteries in these results is the appearance of modular forms of $\mathrm{Sp}(2,\mathbb{Z})$ (or its subgroups) in the expression for the dyon partition function. Since $\mathrm{Sp}(2,\mathbb{Z})$ is the modular group of genus two Riemann surfaces, one might expect that genus two surfaces would play a role in determining the dyon spectrum. The counting that leads to the result however does not explicitly make use of genus two Riemann surfaces [3, 9]. A possible explanation for the role of genus two surfaces has been suggested in [4, 12] by representing the quarter BPS dyon as a string network configuration [29–31] in type IIB string theory on $K3 \times T^2$ and then relating the associated partition function via duality to a configuration of euclidean M5-branes wrapped on $K3$ times a genus two Riemann surface.

Another mystery in this subject is the prescription for computing the spectrum in different domains in the moduli space separated by walls of marginal stability. Naively one would expect that since the dyon spectrum jumps discontinuously across a wall of marginal stability [32–37], the partition function computed in different domains will be different. Instead one finds that as an analytic function of the chemical potentials the partition functions in different domains are identical [9, 11, 13]. However, in order to extract the degeneracies from the partition function in different domains in the moduli space, one needs to choose different contours in the space of complex chemical potentials along which we carry out the Fourier integral of the partition function. A specific set of rules relating the domains in the moduli space and the integration contour in the space of chemical potentials have been given in [13, 17]. The original prescription of [13] restricts the location of the integration contour to be inside a certain region depending on the domain in the moduli space where we want to compute the degeneracy. The value of the integral is independent of the choice of contour as long as the contour lies within this region. This prescription arises from explicit counting of states of quarter BPS dyons [9, 11] and the

JHEP03(2009)151

requirement of S-duality invariance [13, 14]. Ref. [17] proposed a definite choice of contour corresponding to a specific point in the moduli space. While this is consistent with the prescription of [13], currently we do not have any understanding of the physical origin of this prescription. Our main goal in this paper will be to derive this prescription using the relation between the dyon spectrum and genus two surfaces, and in that process make the approach advocated in [4, 12] a little more precise. For simplicity we work with the specific example of heterotic string theory on T^6, but the results should be easily generalizable to other $\mathcal{N} = 4$ supersymmetric string theories.

The rest of the paper is organised as follows. In § 2 we review some of the necessary background material and summarize our main results. In § 3 we describe a specific set of quarter BPS dyon configurations in heterotic string theory on T^6, introduce (real) chemical potentials dual to appropriate charges and relate these chemical potentials to background values of appropriate components of 2-form fields in the theory. We then show how the dyon partition function associated with these states automatically complexifies the chemical potentials and leads to the correct choice of the integration contour in the space of complex chemical potentials. In § 4 we use the strategy of [4, 12] to relate the original partition function to that of an M5-brane on $K3$ times a genus two surface and show that the moduli of the associated genus two surface are given precisely by the complexified chemical potentials which arise in the analysis of § 3. However our analysis does not lead to a foolproof derivation of the actual partition function. Some of the problems were already discussed in [12]; we discuss some additional subtleties in § 5.

Finally we would like to note that the analysis of this paper can in principle be extended to 1/8 BPS states in type IIB string theory on $T^4 \times T^2$ by representing these states as a network of (p, q) 5-branes along T^4 times cycles of T^2. This suggests that this partition function can also be represented as an appropriate quantity associated with a genus two Riemann surface.

2 Background and summary of results

In this section we shall review some material which will be needed for our analysis, and then summarize our results. We shall restrict our analysis to the simplest $\mathcal{N} = 4$ supersymmetric string theory, namely heterotic string theory compactified on T^6 or equivalently type IIA or IIB string theory compactified on $K3 \times T^2$. This theory has 28 U(1) gauge fields and as a result a dyon is characterized by a pair of 28 dimensional vectors (Q, P) labelling electric and magnetic charges. The T-duality group of the theory is a discrete subgroup of $O(6, 22)$ with Q and P transforming in the vector representation of $O(6, 22)$. We denote by L the signature $(6, 22)$ matrix that remains invariant under the $O(6, 22)$ transformation and define

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P \,. \tag{2.1}$$

These are the only independent invariants of the continuous $O(6, 22)$ group which can be constructed out of Q and P. However since only a discrete subgroup of $O(6, 22)$ is a symmetry of string theory, there are other invariants of this discrete group on which

JHEP03(2009)151

physical quantities can depend. $\gcd(Q\wedge P)$ is one such invariant [14] We shall restrict to a subset of dyons for which

$$\gcd(Q\wedge P)=1\,. \tag{2.2}$$

One can show that within this subclass $(Q^2,P^2,Q\cdot P)$ are the complete set of T-duality invariants, i.e. two dyon charges satisfying (2.2) and having same values of Q^2, P^2 and $Q\cdot P$ can be related by a T-duality transformation [22].

We shall denote by $d(Q,P)$ the sixth helicity trace index B_6 of dyons of charge (Q,P) [38]. This effectively counts the number of quarter BPS supermultiplets carrying charge (Q,P), with sign $+1$ (-1) if the average helicity of all the states in the supermultiplet is an integer (integer plus half). This is a protected index and hence is not expected to change under a continuous variation of the moduli of the theory. However there are walls of marginal stability on which a quarter BPS dyon can break up into a pair of half-BPS dyons, and as we cross such a wall in the moduli space the index $d(Q,P)$ can jump. A simple way to label the walls of marginal stability is as follows. One can show that for dyons carrying charge vector of the form given in (2.2) the decay of a quarter BPS dyon into half BPS dyons can take place on a codimension one subspace only if the decay is of the form [18]:

$$(Q,P)\to(\alpha Q+\beta P,\gamma Q+\delta P)+(\delta Q-\beta P,-\gamma Q+\alpha P),\quad \alpha+\delta=1,\quad \alpha\delta=\beta\gamma\,, \tag{2.3}$$

where $(\alpha,\beta,\gamma,\delta)$ are integers. Given $(\alpha,\beta,\gamma,\delta)$, the corresponding wall in the moduli space may be determined by solving the equation

$$m(Q,P)=m(\alpha Q+\beta P,\gamma Q+\delta P)+m(\delta Q-\beta P,-\gamma Q+\alpha P)\,, \tag{2.4}$$

where $m(Q,P)$ denotes the BPS mass of the state of charge (Q,P). Thus for given (Q,P), specifying $(\alpha,\beta,\gamma,\delta)$ determines the wall uniquely, and we can label a wall by the set of integers $(\alpha,\beta,\gamma,\delta)$.

For a given charge (Q,P), these walls of marginal stability divide the moduli space of vacua into different domains. A domain bounded by a set of walls can then be specified by giving the values of $(\alpha,\beta,\gamma,\delta)$ for each of the walls bordering the domain. We shall denote the collection of $(\alpha,\beta,\gamma,\delta)$ labelling a domain by $\vec{c}$. Thus $d(Q,P)$ depends not only on the T-duality invariants Q^2, P^2 and $Q\cdot P$, but also on the domain label $\vec{c}$. Let us denote this function by $f(Q^2/2,P^2/2,Q\cdot P;\vec{c})$, and define the partition function:

$$Z(\rho,\sigma,v;\vec{c})\equiv\sum_{Q^2,P^2,Q\cdot P}(-1)^{Q\cdot P+1}\exp\left\{2\pi i\left(\sigma\,\frac{Q^2}{2}+\rho\,\frac{P^2}{2}+v\,Q\cdot P\right)\right\}\times \tag{2.5}$$
$$\times f(Q^2/2,P^2/2,Q\cdot P;\vec{c})\,.$$

Since the dyon degeneracy function f grows rapidly for large charges, the sum given above is not convergent for real ρ, σ, v, — we need to define it in the complex (ρ,σ,v) space. Let us take

$$\rho=\rho_1+i\rho_2,\qquad \sigma=\sigma_1+i\sigma_2,\qquad v=v_1+iv_2\,. \tag{2.6}$$

In turns out that the sum converges only inside a certain region in the (ρ_2, σ_2, v_2) space. We shall refer to these regions as 'chambers', — whereas the word 'domain' will be reserved for the regions in the moduli space of vacua bounded by walls of marginal stability. This chamber depends on the domain label $\vec{c}$ as follows. Suppose $(\alpha, \beta, \gamma, \delta)$ are the parameters associated with a particular wall. We associate with this a plane in the (ρ_2, σ_2, v_2) space given by

$$\rho_2\gamma - \sigma_2\beta + v_2(\alpha - \delta) = 0\,. \tag{2.7}$$

Now consider the collection of such planes for all $(\alpha, \beta, \gamma, \delta)$ associated with the walls of a particular domain. These form the boundary of a cone in the (ρ_2, σ_2, v_2) space with its vertex at the origin. The chamber in the (ρ_2, σ_2, v_2) space where the sum (2.5) is convergent consists of points inside this chamber lying sufficiently far away from the origin. In other words if we pick any point (a_1, a_2, a_3) in the interior of this cone and choose

$$(\rho_2, \sigma_2, v_2) = \Lambda(a_1, a_2, a_3), \tag{2.8}$$

then for sufficiently large Λ the sum converges. Once the partition function is defined this way inside a chamber, one can extend it to other regions in the complex (ρ, σ, v) space via analytic continuation. It turns out that the function defined this way is independent of the choice of $\vec{c}$ and is given by the inverse of the Igusa cusp form $\Phi_{10}(\rho, \sigma, v)$:

$$Z(\rho, \sigma, v; \vec{c}) = \frac{1}{\Phi_{10}(\rho, \sigma, v)}\,. \tag{2.9}$$

This allows us to invert (2.5) to write

$$d(Q, P) = (-1)^{Q\cdot P+1} \int_{\mathcal{C}(\vec{c})} d\rho d\sigma dv\, e^{-i\pi(\sigma Q^2 + \rho P^2 + 2vQ\cdot P)}\, \frac{1}{\Phi_{10}(\rho, \sigma, v)}\,, \tag{2.10}$$

where the choice of the contour $\mathcal{C}(\vec{c})$ is given by

$$0 \le \rho_1 \le 1, \quad 0 \le \sigma_1 \le 1, \quad 0 \le v_1 \le 1, \quad \rho_2 = M_1, \quad \sigma_2 = M_2, \quad v_2 = M_3\,, \tag{2.11}$$

M_1, M_2, M_3 being constants lying inside the chamber where the original sum is convergent. The result (2.10) was derived by working in a given domain of the moduli space where the type IIB string coupling is weak [9, 11], and then extending the result to other domains by S-duality transformation [13, 14].

A simple prescription for the choice of (ρ_2, σ_2, v_2) that satisfies the requirement of lying inside a given chamber when the moduli lie inside a given domain was given in [17]. Heterotic string theory on T^6 contains a complex scalar modulus τ labelling the axion-dilaton field and another set of 132 real moduli labelling the coset $O(6, 22)/(O(6) \times O(22))$. The latter are parametrized by a symmetric 28×28 matrix M satisfying $M^T L M = L$. We define

$$Q_R^2 = Q^T(M + L)Q, \quad P_R^2 = P^T(M + L)P, \quad Q_R \cdot P_R = Q^T(M + L)P\,. \tag{2.12}$$

Then if we choose[1]

$$\rho_2 = \Lambda \left\{ \frac{|\tau|^2}{\tau_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right\},$$
$$\sigma_2 = \Lambda \left\{ \frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right\},$$
$$v_2 = -\Lambda \left\{ \frac{\tau_1}{\tau_2} + \frac{Q_R \cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}} \right\}, \tag{2.13}$$

then for sufficiently large Λ, (ρ_2, σ_2, v_2) automatically lie inside the correct chamber associated with the domain in which the point (τ, M) lie. This formula picks a given ray inside the cone bounded by the surfaces (2.7). As far as the prescription for the contour is concerned, such precise specification is not necessary; any other ray inside the cone would have been an equally good choice. Nevertheless this formula has some remarkable properties. It correctly takes us from one chamber to another as the moduli cross a wall of marginal stability. Furthermore this formula is S-duality covariant; if we pick another point in the moduli space related to the original one by an S-duality transformation, the ρ_2, σ_2, v_2 given in (2.13) transform correctly so as to preserve the exponent in (2.10). This makes one feel that there must be some deeper origin of this formula that would also naturally explain the correlation between the chambers in the (ρ_2, σ_2, v_2) space and domains in the moduli space without having to make use of S-duality transformation.

However already at this stage we can anticipate a possible difficulty in deriving (2.13). In defining the dyon partition function via (2.5) we need to sum over different charges at *fixed values of* (ρ, σ, v). On the other hand (2.13) determines the imaginary parts of (ρ, σ, v) as a function of the charges and moduli. Thus if we keep the moduli fixed, it would seem that we need to keep changing the imaginary parts of (ρ, σ, v) as we sum over different charges.[2] How can we satisfy these two contradictory requirements? We shall solve this problem by working in a specific corner of the moduli space and with a specific family of dyon charges such that as we vary the charges to generate different $(Q^2, P^2, Q \cdot P)$, the values of (ρ_2, σ_2, v_2) computed from (2.13) remain unchanged. In this case we do not have any difficulty in defining the dyon partition function at fixed values of the moduli.

Like the contour prescription (2.13), the appearance of the Igusa cusp form Φ_{10} in the expression for the dyon partition function is also quite mysterious. The same cusp form also appears in the expression for the two loop partition function of the bosonic string theory. This would lead one to suspect that there is an underlying genus two surface behind the formula given in (2.10). There is however no sign of such a genus two surface in the counting that leads to (2.10); the final result just happens to have this specific form. In a pioneering work, Gaiotto [4] and later Dabholkar and Gaiotto [12] suggested a possible

[1] Throughout this paper we shall use the convention of [24].

[2] In contrast if we keep the label $\vec{c} = \{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}$ fixed, then there is no problem of this kind since the restriction on (ρ_2, σ_2, v_2) to lie inside the cone bounded by the surfaces (2.7) is independent of the charges.

JHEP03(2009)151

origin of this genus two surface from a different viewpoint. The main idea of [4, 12] was to represent the quarter BPS configuration in heterotic string theory on T^6 as a network of strings in a dual type IIB string theory on $K3 \times T^2$. The strings are obtained by wrapping (p, q) five-branes on $K3$, and the network of such strings lie along the plane of the T^2. The partition function of such strings (with $(-1)^F$ inserted in the trace and the integration over the fermionic and the bosonic zero modes associated with the center of mass degrees of freedom factored out) can be represented as a path integral over an euclidean type IIB string theory with periodic boundary condition along the thermal circle. By making a T-duality transformation along the thermal circle and then identifying the resulting type IIA string theory as M-theory on another circle, the partition function can be regarded as that of an euclidean M5-brane wrapped on $K3$ times a genus two surface embedded in T^4. This genus two surface was identified as the origin of the Φ_{10} in the partition function of quarter BPS states.

Our main purpose in this paper is to make this procedure a little more precise and in that process recover the correct prescription for the integration contour as given in (2.13). For this we begin with a configuration in IIB on $K3 \times T^2$ with D5 and NS5-branes wrapped on $K3$ times two different cycles of T^2 and also D strings and fundamental strings wrapped on various cycles of T^2. For fixed D5 and NS5-brane charges Q^2, P^2 and $Q \cdot P$ are given by appropriate linear combinations of the D-string and fundamental string charges, and hence the chemical potentials dual to $Q^2/2$, $P^2/2$ and $Q \cdot P$ can be interpreted as background values of the 2-form fields with one leg along the time direction and another leg along the cycles of T^2. We denote these background fields by σ_1, ρ_1 and v_1 respectively. We then euclideanize the time circle and compactify it on a circle of period $2\pi\beta$ as in [4, 12], with perodic boundary condition on the fermions. The euclidean path integral in the presence of such a background may be represented as a trace of $(-1)^F e^{-2\pi\beta H}$ with an extra insertion of the $\exp\left[2\pi i\left(\rho_1 \frac{P^2}{2} + \sigma_1 \frac{Q^2}{2} + v_1 Q \cdot P\right)\right]$ representing the effect of the background 2-form fields. Furthermore we do not need any extra damping factor for regulating the trace; the damping is provided by the $\exp(-2\pi\beta\, m(Q,P))$ term that appears naturally in the trace, $m(Q,P)$ being the mass of the BPS state of charge (Q,P). By expanding $\exp(-2\pi\beta\, m(Q,P))$ in appropriate limit where the 5-branes give the dominant contribution to $m(Q,P)$ we find that it effectively provides us with a damping factor of $\exp\left\{-2\pi\left(\rho_2 \frac{P^2}{2} + \sigma_2 \frac{Q^2}{2} + v_2 Q \cdot P\right)\right\}$ with (ρ_2, σ_2, v_2) given in (2.13). Thus this procedure automatically leads to the choice of (ρ_2, σ_2, v_2) that makes the partition function convergent.

We then go ahead and follow the prescription of [4, 12] to map the euclidean IIB theory to euclidean M-theory on $K3 \times T^4$ and the string network configuration to a configuration of euclidean M5-brane wrapped on $K3$ times a genus two surface embedded in T^4. Standard duality transformation laws determine the geometry of the final T^4. This in turn allows us to find the moduli of the genus two surface by requiring that the surface is holomorphically embedded in T^4. The result is that the period matrix of the genus two surface is given by

$$\Omega = \begin{pmatrix} \sigma_1 + i\sigma_2 & v_1 + iv_2 \\ v_1 + iv_2 & \rho_1 + i\rho_2 \end{pmatrix} , \tag{2.14}$$

with (ρ_1, σ_1, v_1) determined by the background 2-form fields in the original theory, and (ρ_2, σ_2, v_2) given by (2.13). This allows us the relate the variables (ρ, σ, v) appearing in the definition of the partition function to the moduli of genus two surfaces.

While our analysis tells us that the partition function of quarter BPS states is given by an appropriate partition function on the genus two surface with moduli (ρ, σ, v), we have not explicily computed the partition function and shown that it is given by the inverse of the Igusa cusp form. Refs. [4, 12] already made some progress in this direction, but some subtle points involving fermion zero modes are yet to be sorted out. We point out in § 5 some additional issues in the analysis of the partition function. We hope to return to these points in future.

3 Dyon partition function from 5-brane 1-brane system

We begin with type IIB string theory compactified on $K3 \times S^1 \times \widetilde{S}^1$. By making a T-duality transformation on the circle $\widetilde{S}^1$ to map this to type IIA string theory on $K3 \times S^1 \times \widehat{S}^1$ and then using the duality between type IIA string theory on K3 and heterotic string theory on T^4, we can map this theory to heterotic string theory on $T^4 \times S^1 \times \widehat{S}^1$.[3] Under this duality states in the IIB theory carrying winding charge along S^1 get mapped to magnetically charged states in the heterotic string theory and the states carrying winding charge along $\widetilde{S}^1$ get mapped to electrically charged states in the heterotic string theory. Furthermore the complex structure modulus $-\tau_1 + i\tau_2$ of the $S^1 \times \widetilde{S}^1$ torus, with the $\widetilde{S}^1$ regarded as the a-cycle, gets mapped to the axion-dilaton modulus $\tau \equiv \tau_1 + i\tau_2$ of the dual heterotic string. The other moduli of the IIB theory get mapped to the $O(6,22)/(O(6) \times O(22))$ moduli M of the heterotic string theory, but we shall not need to know the explicit form of this map.

Consider a state in the IIB description containing a (p_1, q_1) 5-brane wrapped on $K3 \times \widetilde{S}^1$ and a (p_2, q_2) 5-brane wrapped on $K3 \times S^1$. If we regard the K3-wrapped 5-branes as strings then such a configuration forms a network of strings on $S^1 \times \widetilde{S}^1$ [4], and the BPS mass of this object, measured in type IIB metric, is given by [31]:[4]

$$m^2_{\rm IIB} = A\,(V_{K3})^2\,\lambda_2^3\,\begin{pmatrix} p_1 & q_1 & p_2 & q_2 \end{pmatrix}(M_0 \pm L_0)\begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix} \tag{3.1}$$

where A denotes the area of $S^1 \times \widetilde{S}^1$ and V_{K3} is the volume of $K3$ measured in the type

[3] This is a slightly different set of duality transformations compared to those used e.g. in [9].

[4] In our convention a (p, q) five brane carries p units of D5-brane charge and q units of NS 5-brane charge. Ref. [31] gives the mass formula for the D-string - fundamental string system in the ten dimensional canonical metric; we have rescaled it by an appropriate factor to convert it to type IIB metric and also to take into account extra factors of string coupling and the volume of $K3$ which appear in the expression for the (p, q) five brane mass compared to the (p, q) string mass.

JHEP03(2009)151

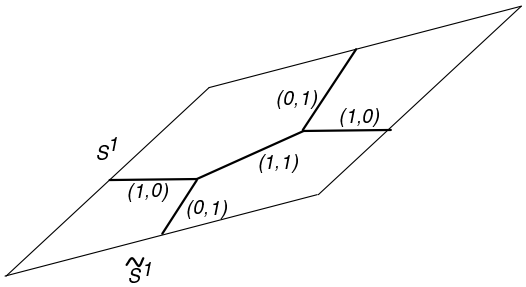

Figure 1. The string network configuration.

IIB metric, and

$$L_0 \equiv \begin{pmatrix} 0 & \mathcal{L} \\ -\mathcal{L} & 0 \end{pmatrix}, \qquad \mathcal{L} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.2}$$

$$M_0 \equiv \frac{1}{\tau_2} \begin{pmatrix} \mathcal{M} & -\tau_1 \mathcal{M} \\ -\tau_1 \mathcal{M} & |\tau|^2 \mathcal{M} \end{pmatrix}, \qquad \mathcal{M} \equiv \frac{1}{\lambda_2} \begin{pmatrix} 1 & -\lambda_1 \\ -\lambda_1 & |\lambda|^2 \end{pmatrix}. \tag{3.3}$$

Here $\lambda = \lambda_1 + i\lambda_2$ denotes the axion-dilaton modulus of the ten dimensional type IIB string theory. The sign in front of L_0 in (3.1) is to be chosen so that the contribution of this term to mass^2 is positive. On the other hand for most general set of charges (Q, P) in the heterotic description, the BPS mass formula measured in the four dimensional canonical metric takes the form

$$m_{\rm can}(Q,P)^2 = \left\{ \frac{1}{\tau_2} |Q_R - \tau P_R|^2 + 2\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2} \right\}, \tag{3.4}$$

where Q_R^2, P_R^2 and $Q_R \cdot P_R$ have been defined in (2.12). Since the canonical four dimensional metric $g_{\mu\nu}$ and the type IIB metric $g^{\rm IIB}_{\mu\nu}$ are related by

$$g_{\mu\nu} = V_{K3}\, A\, \lambda_2^2\, g^{\rm IIB}_{\mu\nu}, \tag{3.5}$$

the BPS mass^2 measured in the type IIB metric takes the form

$$m(Q,P)^2 = V_{K3}\, A\, \lambda_2^2 \left\{ \frac{1}{\tau_2} |Q_R - \tau P_R|^2 + 2\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2} \right\}. \tag{3.6}$$

We now consider a configuration consisting of a $(1,0)$ 5-brane, i.e. a D5-brane, wrapped on $K3 \times \widetilde{S}^1$ and a (0,1) 5-brane, i.e. an NS5-brane, wrapped on $K3 \times S^1$. Thus we have $(p_1, q_1) = (1,0)$ and $(p_2, q_2) = (0,1)$. Since the (1,0) brane wraps $\widetilde{S}^1$ it represents an electric charge vector Q_0 in the dual heterotic string theory. On the other hand the $(0,1)$ brane being wrapped on S^1 represents a magnetic charge P_0. The combined configuration may be represented as a string network on $S^1 \times \widetilde{S}^1$ as shown in figure 1. Our first task will be to express the combinations Q_{0R}^2, P_{0R}^2 and $Q_{0R} \cdot P_{0R}$ in terms of type IIB variables. This is done by comparing the BPS mass formulæ (3.1), (3.6) applied to the charge vectors

JHEP03(2009)151

$(Q_0, 0)$, $(0, P_0)$ and (Q_0, P_0). We get[5]

$$\begin{aligned}
m(Q_0,0)^2 &= V_{K3}\, A\, \lambda_2^2\, \frac{Q_{0R}^2}{\tau_2} = A\,(V_{K3})^2\, \lambda_2^3\, \frac{1}{\lambda_2\tau_2}\,, \\
m(0,P_0)^2 &= V_{K3}\, A\, \lambda_2^2\, \frac{P_{0R}^2|\tau|^2}{\tau_2} = A\,(V_{K3})^2\, \lambda_2^3\, \frac{|\tau|^2|\lambda|^2}{\lambda_2\tau_2}\,, \\
m(Q_0,P_0)^2 &= V_{K3}\, A\, \lambda_2^2\, \left\{\frac{1}{\tau_2}|Q_{0R}-\tau P_{0R}|^2 + 2\sqrt{Q_{0R}^2P_{0R}^2-(Q_{0R}\cdot P_{0R})^2}\right\} \\
&= A\,(V_{K3})^2\, \lambda_2^3\, \left[\frac{1}{\lambda_2\tau_2} + \frac{|\tau|^2|\lambda|^2}{\lambda_2\tau_2} + 2\frac{\tau_1\lambda_1}{\lambda_2\tau_2} + 2\right]\,.
\end{aligned} \tag{3.7}$$

This gives

$$\begin{aligned}
Q_{0R}^2 &= V_{K3}, \\
P_{0R}^2 &= V_{K3}\,|\lambda|^2, \\
Q_{0R}\cdot P_{0R} &= -V_{K3}\,\lambda_1\,.
\end{aligned} \tag{3.8}$$

We now add to the previous system n_1 units of fundamental string charge wrapped on $\widetilde{S}^1$, n_2 units of fundamental string charge wrapped on S^1, m_1 units of D-string charge wrapped on $\widetilde{S}^1$ and m_2 units of D-string charge wrapped on S^1. These can be regarded as excitations of the original D5-brane - NS5-brane system involving small instantons and world-volume electric fields. Unbroken supersymmetry imposes the constraints $m_1 \geq -1$, $n_2 \geq -1$ after taking into account the induced D-string and the fundamental string charges on the $K3$ wrapped D5 and NS5-branes. Then by following the standard duality chain to map these charges into the heterotic description, we get

$$\begin{aligned}
Q^2 &\equiv Q^T L Q = 2m_1, \\
P^2 &\equiv P^T L P = 2n_2, \\
Q\cdot P &\equiv Q^T L P = m_2 + n_1\,.
\end{aligned} \tag{3.9}$$

The expression for the mass $m(Q,P)$ gets modified in the presence of these charges. We shall assume that V_{K3} is large so that the 5-branes still give the dominant contribution to $m(Q,P)$ and compute the first order correction to $m(Q,P)$ due to the D1 brane and fundamental string charges. It is easy to see that in eq. (3.6) this contribution comes from the correction to Q_R^2, P_R^2 and $Q_R\cdot P_R$ given in (2.12) from the $Q^2 = 2m_1$, $P^2 = 2n_2$ and $Q\cdot P = m_2 + n_1$ terms respectively, — corrections to $Q^T M Q$, $P^T M P$ and $Q^T M P$ terms

[5] A D5-brane wrapped on $K3$ carries 1 unit of D1-brane charge and an NS 5-brane wrapped on $K3$ carries 1 unit of fundamental string charge. In (3.7), (3.8) we have assumed that these 1-brane charges have been neutralized by appropriate configuration of D-string and fundamental string.

JHEP03(2009)151

are suppressed by further powers of V_{K3}.[6] This gives

$$\begin{aligned}
\delta m(Q,P) &= \frac{1}{m(Q,P)}\, V_{K3}\, A\, \lambda_2^2 \left[\frac{Q^2}{2}\left\{\frac{1}{\tau_2}+\frac{P_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}\right. \\
&\quad \left. +\frac{P^2}{2}\left\{\frac{|\tau|^2}{\tau_2}+\frac{Q_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}-Q\cdot P\left\{\frac{\tau_1}{\tau_2}+\frac{Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}\right] \\
&= A^{1/2}\,\lambda_2^{1/2}\left[\frac{1}{\lambda_2\tau_2}+\frac{|\tau|^2|\lambda|^2}{\lambda_2\tau_2}+2\frac{\tau_1\lambda_1}{\lambda_2\tau_2}+2\right]^{-1/2} \\
&\quad \left[\frac{Q^2}{2}\left\{\frac{1}{\tau_2}+\frac{P_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}+\frac{P^2}{2}\left\{\frac{|\tau|^2}{\tau_2}+\frac{Q_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}\right. \\
&\quad \left. -Q\cdot P\left\{\frac{\tau_1}{\tau_2}+\frac{Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}\right] \qquad (3.10)
\end{aligned}$$

where in the last step we have used the leading order expression for $m(Q,P)$ given in the last line of (3.7). Note that on the right hand side of this equation we have used the arguments Q, P instead of Q_0, P_0. Since the difference between $(Q_R^2, P_R^2, Q_R\cdot P_R)$ and $(Q_{0R}^2, P_{0R}^2, Q_{0R}\cdot P_{0R})$ is already of the first order, the error due to the replacement of (Q_0,P_0) by (Q,P) on the right hand side is of higher order. We shall work in the limit $V_{K3}\to\infty$ at fixed λ, τ and A; in this limit (3.10) is the exact expression for $\delta m(Q,P)$.

We denote by y and $\widetilde{y}$ the coordinates along S^1 and $\widetilde{S}^1$ respectively, and by t the time coordinate. Let $C^{(2)}$ and $B^{(2)}$ denote the RR and NSNS 2-form fields. We now make a Wick rotation $t\to -i\tau$, compactify the τ coordinate on a circle of period $2\pi\,\beta$ with periodic boundary condition along the circle, and switch on background values of $C^{(2)}_{\tau y}$, $C^{(2)}_{\tau\widetilde{y}}$, $B^{(2)}_{\tau y}$ and $B^{(2)}_{\tau\widetilde{y}}$ of the form

$$C^{(2)}_{\tau y}=B^{(2)}_{\tau\widetilde{y}}=v_1,\quad C^{(2)}_{\tau\widetilde{y}}=\sigma_1,\quad B^{(2)}_{\tau y}=\rho_1\,, \qquad (3.11)$$

all normalized so that ρ_1, σ_1 and v_1 have period 1. Since the background fields $C^{(2)}_{\tau\widetilde{y}}$, $C^{(2)}_{\tau y}$, $B^{(2)}_{\tau\widetilde{y}}$ and $B^{(2)}_{\tau y}$ couple to the charges m_1, m_2, n_1 and n_2 respectively, the presence of the background (3.11) corresponds to inserting a factor of[7]

$$\exp\left[2\pi i\left(C^{(2)}_{\tau\widetilde{y}}m_1+C^{(2)}_{\tau y}m_2+B^{(2)}_{\tau\widetilde{y}}n_1+B^{(2)}_{\tau y}n_2\right)\right]=\exp\left[2\pi i\left(\rho_1\frac{P^2}{2}+\sigma_1\frac{Q^2}{2}+v_1 Q\cdot P\right)\right], \qquad (3.12)$$

into the functional integral. In the Hamiltonian formulation this functional integral may be represented as a trace with an additional insertion of $e^{-2\pi\beta H}(-1)^F$.[8] Identifying H as

[6] This requires taking $V_{K3}\to\infty$ limit in such a way that the off-diagonal components of M which couple the wrapped 5-brane charges to the wrapped 1-brane charges vanish in this limit.

[7] In our convention $(C^{(2)}_{MN}, B^{(2)}_{MN})$ and $(-n_i, m_i)$ transform as SL$(2,R)$ doublets. Thus the exponent given in (3.12) is S-duality invariant.

[8] The insertion of $(-1)^F$ into the trace reflects the effect of putting periodic boundary condition on the fermions along the τ direction. We shall assume as in [4, 12, 37] that the trace over the center of

the mass of the brane configuration and removing the contribution to H from the leading term $m(Q_0, P_0)$ that does not depend on m_i, n_i, we see that the path integral computes the quantity

$$\begin{aligned} & Tr\left[(-1)^F \exp\left\{2\pi i\left(\rho_1\frac{P^2}{2}+\sigma_1\frac{Q^2}{2}+v_1 Q\cdot P\right)\right\}\exp\left\{-2\pi\beta\delta m(Q,P)\right\}\right] \\ & \quad = Tr\left[(-1)^F \exp\left\{2\pi i\left(\rho_1\frac{P^2}{2}+\sigma_1\frac{Q^2}{2}+v_1 Q\cdot P\right)\right\}\times\right. \\ & \qquad \left.\times\exp\left\{-2\pi\left(\rho_2\frac{P^2}{2}+\sigma_2\frac{Q^2}{2}+v_2\, Q\cdot P\right)\right\}\right] \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \rho_2 &= \Lambda\left\{\frac{|\tau|^2}{\tau_2}+\frac{Q_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\} \\ \sigma_2 &= \Lambda\left\{\frac{1}{\tau_2}+\frac{P_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\} \\ v_2 &= -\Lambda\left\{\frac{\tau_1}{\tau_2}+\frac{Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right\}, \end{aligned} \tag{3.14}$$

$$\Lambda \equiv \beta\, A^{1/2}\,\lambda_2^{1/2}\left[\frac{1}{\lambda_2\tau_2}+\frac{|\tau|^2|\lambda|^2}{\lambda_2\tau_2}+2\frac{\tau_1\lambda_1}{\lambda_2\tau_2}+2\right]^{-1/2}. \tag{3.15}$$

In going from the first to the second line of (3.13) we have used the expression for $\delta m(Q,P)$ given in (3.10). We shall take $\beta A^{1/2}$ large but finite so as to provide sufficient exponential suppression factor and make the trace finite. Defining

$$\rho \equiv \rho_1 + i\rho_2, \quad \sigma \equiv \sigma_1 + i\sigma_2, \quad v \equiv v_1 + iv_2\,, \tag{3.16}$$

we can express (3.13) as

$$Tr\left[(-1)^F \exp\left\{2\pi i\left(\rho\frac{P^2}{2}+\sigma\frac{Q^2}{2}+v\, Q\cdot P\right)\right\}\right]. \tag{3.17}$$

Thus we see that the path integral automatically leads to the dyon partition function with complex (ρ,σ,v), with the imaginary parts of (ρ,σ,v) given by the prescription of [17] given in (2.13). Due to the $(-1)^F$ insertion in the part integral the trace will include sum over BPS states only. We note however that in order to relate (3.17) to the dyon partition function defined in (2.5) we must insert additional projection operators into the trace in (3.17) which restrict the states over which we sum. This is because a given set of values of $(Q^2, P^2, Q\cdot P)$ may arise from many different charge vectors and in defining

mass degrees of freedom and their fermionic superpartners have been factorized; otherwise trace over these additional zero modes will make the result vanish, and we need to insert six powers of helicity into the trace to get a non-zero answer.

JHEP03(2009)151

the partition function (2.5) we have summed over distinct values of $(Q^2, P^2, Q\cdot P)$ instead of over all charges. For example (3.9) shows that $Q\cdot P$ depends only on the combination m_2+n_1; thus for fixed Q^2, P^2 if we want to count each value of $Q\cdot P$ only once we must not perform independent sums over m_2 and n_1. We have also assumed implicitly that we are summing over states which do not carry any charges associated with a 3-brane wrapped on a 2-cycle of $K3$ and a 1-cycle of $S^1\times\widetilde{S}^1$. Finally, we have restricted the sum over states to the sector with zero spatial momentum both along the circles S^1 and $\widetilde{S}^1$ and along the non-compact directions; this requires insertion of yet more projection operators. We shall not keep track of these projection operators in subsequent analysis; these will be important in explicit computation of the partition function but not in finding the physical interpretation of (ρ,σ,v). We shall return to this issue in § 5.

In § 4 we shall find a geometric description of this partition function following the duality maps given in [4, 12].

4 Chemical potentials to period matrix

In this section we shall find a geometric interpretation of the dyon partition function introduced in the previous section. For this we need to use a dual M-theory description of the theory. We first make a T-duality transformation along the euclidean time circle to map the euclidean type IIB string theory described in the previous section to euclidean type IIA string theory compactified on $K3\times S^1\times\widetilde{S}^1\times S^1_T$, – S^1_T being the circle dual to the euclidean time circle of IIB. This in turn can now be regarded as euclidean M-theory on $K3\times S^1\times\widetilde{S}^1\times S^1_T\times S^1_M$. Following the chain of dualities we can find the relationship between the parameters labelling the M-theory torus $T^4\equiv S^1\times\widetilde{S}^1\times S^1_T\times S^1_M$ and the parameters labelling the original type IIB compactification. In particular one finds that the parameters τ, λ, $C^{(2)}_{\tau y}$, $C^{(2)}_{\tau\widetilde{y}}$, $B^{(2)}_{\tau y}$ and $B^{(2)}_{\tau\widetilde{y}}$ of the type IIB string theory can be regarded as components of the metric along the four torus in the M-theory description. Alternatively we can represent the M-theory torus as euclidean space with a standard metric

$$(dx^1)^2+(dx^2)^2+(dy^1)^2+(dy^2)^2\,, \tag{4.1}$$

modded out by a lattice Λ. In this case the information about τ, λ, $C^{(2)}_{\tau y}$, $C^{(2)}_{\tau\widetilde{y}}$, $B^{(2)}_{\tau y}$ and $B^{(2)}_{\tau\widetilde{y}}$ is encoded in the generators of the lattice Λ.

First consider the case when all the off-diagonal fields, e.g. τ_1, λ_1, $C^{(2)}_{ij}$ and $B^{(2)}_{ij}$ vanish. In this case T^4 is a direct product of four circles. Let x^1, x^2, y^1 and y^2 denote coordinates along the circles S^1_M, S^1_T, $\widetilde{S}^1$ and S^1 respectively. Standard duality transformation rules then tell us that these circles have periods L_2, $L_2\lambda_2$, L_1 and $L_1\tau_2$ respectively for

$$L_1=2\pi\,\beta^{1/3}\lambda_2^{1/3}A^{1/2}\tau_2^{-1/2},\qquad L_2=2\pi\,\lambda_2^{-2/3}\,\beta^{-2/3}\,. \tag{4.2}$$

This is associated with a lattice Λ generated by the unit vectors

$$\begin{pmatrix}L_2\\0\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\L_2\lambda_2\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\L_1\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\0\\L_1\tau_2\end{pmatrix}. \tag{4.3}$$

JHEP03(2009)151

Furthermore the volume of $K3$ in the final M-theory metric is related to that measured in the original IIB metric via the relation

$$V_{K3}^M = \beta^{4/3}\,\lambda_2^{4/3}\,V_{K3}\,. \tag{4.4}$$

As mentioned above, deformations associated with the parameters τ_1, λ_1, $C_{ij}^{(2)}$ and $B_{ij}^{(2)}$ can be shown to be associated with the geometric deformation of the four torus spanned by S^1, $\widetilde{S}^1$, S_M^1 and S_T^1. One can find the parameters of the deformed torus in terms of the parameters of the type IIB theory. The four generators of the lattice Λ associated with this deformed M-theory torus turn out to be

$$e_1 = \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -L_2\lambda_1 \\ L_2\lambda_2 \\ 0 \\ 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} L_2 C^{(2)}_{\tau\widetilde{y}} - L_2\lambda_1 B^{(2)}_{\tau\widetilde{y}} \\ L_2\lambda_2 B^{(2)}_{\tau\widetilde{y}} \\ L_1 \\ 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} L_2 C^{(2)}_{\tau y} - L_2\lambda_1 B^{(2)}_{\tau y} \\ L_2\lambda_2 B^{(2)}_{\tau y} \\ -L_1\tau_1 \\ L_1\tau_2 \end{pmatrix}. \tag{4.5}$$

Equivalently one can describe the M-theory torus as a product of four circles, each of period 2π, with metric $g_{ij} = e_i \cdot e_j$. Note that a shift of $C^{(2)}_{\tau y}$, $C^{(2)}_{\tau\widetilde{y}}$, $B^{(2)}_{\tau y}$ or $B^{(2)}_{\tau\widetilde{y}}$ by an integer produces an integer linear combination of the original e_i's and generate the same lattice.

Our next task will be to study the fate of the original D5-NS5-brane configuration in the M-theory description. First consider the case where λ_1, τ_1, $C_{ij}^{(2)}$ and $B_{ij}^{(2)}$ vanish. In this case the D5-brane wrapped along $K3 \times \widetilde{S}^1$ becomes an euclidean M5-brane wrapped along $K3 \times \widetilde{S}^1 \times S_M^1$ and the NS5-brane wrapped on $K3 \times S^1$ becomes an euclidean M5-brane wrapped on $K3 \times S^1 \times S_T^1$. Leaving aside the K3 part, this can be regarded as a degenerate genus two surface embedded in T^4, with period matrix

$$\Omega \equiv \begin{pmatrix} \check{\sigma} & \check{v} \\ \check{v} & \check{\rho} \end{pmatrix} = \begin{pmatrix} i\,L_1/L_2 & 0 \\ 0 & i\,L_1\tau_2/(L_2\lambda_2) \end{pmatrix}. \tag{4.6}$$

For more general background in the original type IIB string theory, the M5-brane configuration in the dual M-theory will have the form of $K3$ times a non-singular genus two surface embedded in T^4. Our goal now will be to determine the period matrix of this genus two surface in terms of the parameters labelling the original type IIB background. The main tool will be supersymmetry which requires that the genus two surface is holomorphically embedded in T^4. Let ω_1 and ω_2 be the two linearly independent holomorphic 1-forms on the genus two surface and let A_1, A_2, B_1, B_2 be a basis of integral homology cycles satisfying

$$A_i \cap B_j = \delta_{ij}, \quad A_i \cap A_j = B_i \cap B_j = 0\,. \tag{4.7}$$

Then the requirement that the genus two surface is holomorphically embedded in T^4 amounts to the constraint that the T^4 is given by $\mathbb{C}^2/\overline{\Lambda}$, where the lattice $\overline{\Lambda}$ is gener-

ated by the complex two dimensional vectors [39]

$$\bar{e}_1 = \begin{pmatrix} \oint_{A_1} \omega_1 \\ \oint_{A_1} \omega_2 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} \oint_{A_2} \omega_1 \\ \oint_{A_2} \omega_2 \end{pmatrix}, \quad \bar{e}_3 = \begin{pmatrix} \oint_{B_1} \omega_1 \\ \oint_{B_1} \omega_2 \end{pmatrix}, \quad \bar{e}_4 = \begin{pmatrix} \oint_{B_2} \omega_1 \\ \oint_{B_2} \omega_2 \end{pmatrix}. \tag{4.8}$$

$\mathbb{C}^{(2)}$ is endowed with the standard metric $|dz_1|^2 + |dz_2|^2$. Now by taking appropriate complex linear combinations of the two ω_i's we can guarantee that

$$\oint_{A_i} \omega_j = \delta_{ij}, \quad \int_{B_i} \omega_j = \Omega_{ij}\,, \tag{4.9}$$

where $\Omega = \begin{pmatrix} \check{\sigma} & \check{v} \\ \check{v} & \check{\rho} \end{pmatrix}$ is the period matrix of the genus two surface. Thus for a general choice of the ω_i's, related to the one above by a $GL(2,\mathbb{C})$ transformation matrix V, we have

$$\bar{e}_1 = V \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{e}_2 = V \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{e}_3 = V \begin{pmatrix} \check{\sigma} \\ \check{v} \end{pmatrix}, \quad \bar{e}_4 = V \begin{pmatrix} \check{v} \\ \check{\rho} \end{pmatrix}. \tag{4.10}$$

Our goal is to compare, up to SO(4) rotations, the basis vectors $\bar{e}_i$ given in (4.10) with those given in (4.5) to find the relationship between the parameters τ, λ, $C^{(2)}_{ij}$, $B^{(2)}_{ij}$ of the M-theory torus and the parameters $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ labelling the moduli of the genus two surface. For this we have to express the two component complex vectors $\bar{e}_i$ given in (4.10) as four component real vectors by separately picking out the real and imaginary parts of $\bar{e}_i$. If $\bar{e}_{iR}$ and $\bar{e}_{iI}$ denote the real and the imaginary parts of $\bar{e}_i$, then the inner product between $\bar{e}_i$ and $\bar{e}_j$, regarded as real vectors, is given by $\bar{e}_{iR}\cdot\bar{e}_{jR} + \bar{e}_{iI}\cdot\bar{e}_{jI} = Re(\bar{e}_i^*\cdot\bar{e}_j)$. Defining a, b, c through

$$V^\dagger V = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}, \quad a, b \in \mathbb{R}, \quad c \in \mathbb{C}\,, \tag{4.11}$$

we get

$$Re(\bar{e}_i^*\cdot\bar{e}_j) = Re\begin{pmatrix} a & c & a\check{\sigma}+c\check{v} & a\check{v}+c\check{\rho} \\ c^* & b & c^*\check{\sigma}+b\check{v} & c^*\check{v}+b\check{\rho} \\ \check{\sigma}^*a+\check{v}^*c^* & \check{\sigma}^*c+\check{v}^*b & \check{\sigma}^*(a\check{\sigma}+c\check{v}) & \check{\sigma}^*(a\check{v}+c\check{\rho}) \\ & & +\check{v}^*(c^*\check{\sigma}+b\check{v}) & +\check{v}^*(c^*\check{v}+b\check{\rho}) \\ \check{v}^*a+\check{\rho}^*c^* & \check{v}^*c+\check{\rho}^*b & \check{v}^*(a\check{\sigma}+c\check{v}) & \check{v}^*(a\check{v}+c\check{\rho}) \\ & & +\check{\rho}^*(c^*\check{\sigma}+b\check{v}) & +\check{\rho}^*(c^*\check{v}+b\check{\rho}) \end{pmatrix}. \tag{4.12}$$

We can now compare this with the inner product matrix $e_i\cdot e_j$ constructed from the basis vectors given in (4.5). In computing $e_i\cdot e_j$ we shall consider the special case where $C^{(2)}_{\tau y} = B^{(2)}_{\tau\tilde{y}}$, since this is the background used in the analysis of § 3. In this case the equations $Re(\bar{e}_i^*\cdot\bar{e}_j) = e_i\cdot e_j$ can be solved to give:

$$\begin{aligned} a = L_2^2, \quad b = L_2^2|\lambda|^2, \quad c = -L_2^2\lambda_1, \quad \check{\sigma}_1 = C^{(2)}_{\tau\tilde{y}}, \quad \check{\rho}_1 = B^{(2)}_{\tau y}, \quad \check{v}_1 = C^{(2)}_{\tau y} = B^{(2)}_{\tau\tilde{y}}, \\ \check{\sigma}_2^2 - 2\lambda_1\check{v}_2\check{\sigma}_2 + |\lambda|^2\check{v}_2^2 = L_1^2/L_2^2\,, \\ \check{\sigma}_2\check{v}_2 - \lambda_1(\check{\sigma}_2\check{\rho}_2 + \check{v}_2^2) + |\lambda|^2\check{v}_2\check{\rho}_2 = -\tau_1 L_1^2/L_2^2\,, \\ \check{v}_2^2 - 2\lambda_1\check{v}_2\check{\rho}_2 + |\lambda|^2\check{\rho}_2^2 = |\tau|^2 L_1^2/L_2^2\,. \end{aligned} \tag{4.13}$$

JHEP03(2009)151

The equations for $\check\sigma_2$, $\check\rho_2$ and $\check v_2$ can be solved as

$$\begin{aligned}
\check\sigma_2 &= \check\Lambda\left\{\frac{|\lambda|^2}{\lambda_2}+\frac{1}{\tau_2}\right\} = \check\Lambda\left\{\frac{1}{\tau_2}+\frac{P_{0R}^2}{\sqrt{Q_{0R}^2P_{0R}^2-(Q_{0R}\cdot P_{0R})^2}}\right\},\\
\check\rho_2 &= \check\Lambda\left\{\frac{1}{\lambda_2}+\frac{|\tau|^2}{\tau_2}\right\} = \check\Lambda\left\{\frac{|\tau|^2}{\tau_2}+\frac{Q_{0R}^2}{\sqrt{Q_{0R}^2P_{0R}^2-(Q_{0R}\cdot P_{0R})^2}}\right\},\\
\check v_2 &= \check\Lambda\left\{\frac{\lambda_1}{\lambda_2}-\frac{\tau_1}{\tau_2}\right\} = -\check\Lambda\left\{\frac{\tau_1}{\tau_2}+\frac{Q_{0R}\cdot P_{0R}}{\sqrt{Q_{0R}^2P_{0R}^2-(Q_{0R}\cdot P_{0R})^2}}\right\},
\end{aligned} \tag{4.14}$$

where we have used eqs. (3.8) and

$$\check\Lambda = L_1L_2^{-1}\,\tau_2\,\left\{1+2\lambda_1\tau_1+2\lambda_2\tau_2+|\lambda|^2|\tau|^2\right\}^{-1/2}. \tag{4.15}$$

Finally let us discuss the effect of inclusion of the D-strings and fundamental strings in the original type IIB description. These correspond to appropriate excitations on the M5-brane world volume. However in the $V_{K3}\to\infty$ limit we expect the effect of these excitations on the M5-brane geometry to vanish and $(\check\rho,\check\sigma,\check v)$ given by (4.13), (4.14) continue to describe the moduli of the genus two surface which the M5-brane wraps. Furthermore in this limit we can replace $(Q_{0R}^2,P_{0R}^2,Q_{0R}\cdot P_{0R})$ by $(Q_R^2,P_R^2,Q_R\cdot P_R)$ in (4.14). Comparing (4.13), (4.14), (4.15) with (3.11), (3.14), (3.15), we now see that

$$\check\Lambda = \Lambda\,, \tag{4.16}$$

and

$$\check\rho=\rho,\quad \check\sigma=\sigma,\quad \check v=v\,. \tag{4.17}$$

This shows that the dyon partition function given in (3.17) is given by an appropriate partition function on a genus two Riemann surface with modular parameters (ρ,σ,v). Furthermore, as already noted below (3.17), the imaginary parts of (ρ,σ,v) are automatically set according to the prescription given in [17].

5 The partition function

While our analysis determines the moduli of the genus two surface on which the M5-brane is wrapped, we have not determined precisely what computation we need to perform on this genus two surface to extract the partition function. Since the low energy world-sheet theory of M5-brane wrapped on K3 coincides with that of a fundamental heterotic string on a transverse T^3 in static gauge Green-Schwarz formulation, one might expect the final partition function to be given by that of an euclidean heterotic string in $T^4\times T^3$ [4, 12].[9] However such a partition function vanishes due to the right-moving fermion zero modes,

[9] Note that in the $V_{K3}^M\to\infty$ limit, describing a K3 wrapped M5-brane as a fundamental heterotic string as in [4, 12] is not useful in general. However for computing an index one may still be able to use this description.

JHEP03(2009)151

and one must pick only the left-moving part of the partition function to get the desired result given by the inverse of the Igusa cusp form. One might expect that this prescription should follow from the fact that in defining the original partition function in the type IIB string theory we have removed by hand the right-moving fermion zero modes, but exactly what this translates to in the M-theory description is not understood [12]. One possibility will be to begin with the helicity trace by inserting six powers of helicity in the original trace in type IIB string theory and then carefully keeping track of this factor during the duality transformations.

Another issue arises from the need to insert additional projection operators into the trace in the original type IIB description so that the trace receives contribution only from a subset of dyon charges. As mentioned below (3.17), this is necessary to ensure that in the trace each value of $(Q^2, P^2, Q\cdot P)$ is counted only once. In particular although the full set of allowed charges consists of 28 electric charges and 28 magnetic charges, the charge configuration we have considered span only a small subspace. Furthermore we have restricted the trace to be over states carrying zero spatial momenta. What do such restrictions correspond to in the M-theory picture? To answer this question we first note that in the original type IIB description the missing charges can be divided into four classes: 1) Kaluza-Klein monopoles associated with the circles S^1 and $\widetilde{S}^1$, 2) more general 5-branes wrapped on $K3$ times S^1 or $\widetilde{S}^1$, 3) charges associated with D3-branes wrapped on a 2-cycle of $K3$ and S^1 or $\widetilde{S}^1$ and 4) momenta along S^1 and $\widetilde{S}^1$. Under the duality map of § 4 that takes the euclidean type IIB string theory to the euclidean M-theory, the Kaluza-Klein monopoles get mapped to Kaluza-Klein monopoles. Since we do not expect the partition function associated with an euclidean M5-brane to include contribution from the Kaluza-Klein monopole sector, it is natural to set the Kaluza-Klein monopole charges in the original type IIB theory to zero. On the other hand a configuration carrying a general set of 5-brane charges compared to the one considered in § 3, e.g. a (p_1, q_1) 5-brane along $K3\times\widetilde{S}^1$ and a (p_2, q_2) 5-brane along $K3\times S^1$, will get mapped to a euclidean M5-brane configuration wrapping $K3$ times a genus 2-surface in T^4, but the embedding of the genus two surface into T^4 will be topologically distinct for different (p_1, q_1, p_2, q_2). Thus the M5-brane partition function, regarded as a partition function on a genus two surface, will naturally include contribution from states with a fixed set of (p_1, q_1, p_2, q_2) which in our case is $(1, 0, 0, 1)$. Charges associated with D3-branes wrapped on a 2-cycle of $K3$ and S^1 or $\widetilde{S}^1$ correspond to switching on magnetic flux on the D5 or NS5-brane world-volume along a 2-cycle of K3. By following the chain of dualities one finds that this corresponds to switching on the flux of the 3-form field strength on the euclidean M5-brane along a 2-cycle of K3 times S^1_M or S^1_T. Thus requiring that the D3-brane charges vanish in the original type IIB theory amounts to restricting the path integral over M5-brane degrees of freedom to sectors with zero 3-form field strength flux through the 2-cycles of $K3$ times S^1_M or S^1_T.[10] Finally the effect of switching on momenta, either along S^1 or $\widetilde{S}^1$ or along

[10] In our convention this corresponds to zero 3-form field strength along the product of a 2-cycle on K3 and one of the A-cycles on the genus two surface. Since the 3-form field strength is self-dual, this effectively restricts all possible flux of the 3-form field strength on the euclidean M5-brane world-volume to zero barring the issue that in an euclidean signature space the requirement of a self-duality makes a 3-form complex.

JHEP03(2009)151

the non-compact directions, corresponds to collective motion of the string network and does not affect either the degeneracy of states or the values of Q^2, P^2 and $Q\cdot P$. Thus on the type IIB side restricting these momenta to zero should correspond to factoring out (or freezing) the euclidean path integral over these collective modes. We need to determine what operation this corresponds to on the M5-brane partition function; we shall come back to this issue shortly.

This does not exhaust the list of restrictions we need to impose on the M5-brane path integral. As mentioned below (3.17), independent sum over m_2 and n_1 gives the same $(Q^2, P^2, Q\cdot P)$ infinite number of times since the latter depends on m_2 and n_1 only through the combination m_2+n_1. Thus charge vectors of the form (n_1+k, m_2-k) generate the same set of invariants as (n_1, m_2). We shall now argue that this sum over k can also be regarded as a sum over momentum conjugate to a collective mode. For this we note that if we just have a D5-brane wrapped on $K3\times\widetilde{S}^1$, then we can generate a fundamental string charge along $\widetilde{S}^1$ by switching on an electric field along $\widetilde{S}^1$. This can be interpreted as the momentum conjugate to the Wilson line along $\widetilde{S}^1$ on the D5-brane, and contributes to the quantum number n_1. Similarly by switching on an electric field on NS 5-brane along $K3\times S^1$ we can generate a D-string charge along S^1. This is the momentum conjugate to the Wilson line along S^1 on the NS 5-brane and contributes to $-m_2$. However once the D5-brane and the NS 5-brane join to form a string network as in figure 1, the Wilson lines on D5-brane and NS 5-branes do not describe independent collective coordinates. To see this we note from figure 1 that in the string network configuration there is an intermediate $(1,1)$ 5-brane, and switching on an electric field along this generates equal but opposite amount of D-string and the fundamental string charge. Thus by requiring conservation of fundamental and D-string charges at the junction we see that the electric fields carried by the D5-brane and the NS 5-brane must be correlated such that the net D-string charge carried along S^1 and the net fundamental string charge carried along $\widetilde{S}^1$ are equal. In other words only changes of the form $(n_1, m_2)\to(n_1+k, m_2-k)$ can be regarded as due to excitations of a collective coordinate. Thus picking one representative (m_2, n_1) for each m_2+n_1 corresponds to freezing this collective degree of freedom. On the other hand fluctuations of the string network which change the values of m_2+n_1 cannot be regarded as collective coordinate excitations.

Thus our task now is to determine what operation on the M-theory side would correspond to freezing or factoring out the contribution from the various collective modes of the string network on the type IIB side. By following the chain of dualities leading from the IIB description to the M-theory description we can identify the collective deformations of the network in the euclidean IIB theory to those of the M5-brane in euclidean M-theory. In particular the collective deformations associated with the translation of the network along the non-compact directions, S^1 and $\widetilde{S}^1$ correspond respectively to the freedom of translating the euclidean M5-brane along the non-compact directions, S^1 and $\widetilde{S}^1$. On the other hand the collective deformation corresponding to the Wilson line on the network corresponds to the freedom of switching on an anti-symmetric 2-form field, proportional to the volume form on the genus two surface, on the M5-brane world-volume. Nevertheless it is not entirely clear what a time dependent collective excitation of the string junction

JHEP03(2009)151

in IIB side translates to in the M-theory side. If ϕ denotes a collective mode of the string network then freezing it would correspond to setting $\phi(\tau) = 0$ in the euclidean path integral. While the zero mode of $\phi(\tau)$ has a simple interpretation in the dual description involving the euclidean M5-brane, the physical interpretation of the τ dependent part of $\phi(\tau)$ in the M5-brane world-volume theory is more complicated since the duality relating the two descriptions involves a T-duality transformation that converts momentum modes along the euclidean time circle into fundamental string winding modes and vice versa.

To summarize, in order that the partition function of the string network in the type IIB description computes the dyon partition function of our interest, we must freeze all the collective excitations of the string network (or factor out their contribution from the partition function), and at the same time set the magnetic flux on the 5-branes through the 2-cycles of K3 to zero. In the M-theory description the latter can be interpreted as setting to zero appropriate flux of 3-form field strength on the M5-brane. However the physical interpretation of freezing the collective coordinates of the string network is not entirely clear since the duality relating the type IIB description and the M-theory description involves T-duality along the euclidean time circle. One could however be a little less ambitious and take the point of view that since on the type IIB side the contribution from the bosonic (fermionic) collective modes do not affect the (ρ, σ, v) dependence of the partition function but only generate an overall divergent (zero) factor, we could begin with the partition function of the euclidean string network without freezing any collective modes and then simply throw away the overall (ρ, σ, v) independent divergent or zero factors. This would translate to a similar operation on the M5-brane partition function. This clearly would not determine the overall normalization of the partition function but could determine its dependence on (ρ, σ, v). Along this line one could also make use of the fact that in the type IIB description the partition function Z depends only on (ρ, σ, v) and not their complex conjugates due to (3.17). Thus in the final answer we can just pick up the holomorphic part of the M5-brane partition function; any dependence on $(\bar{\rho}, \bar{\sigma}, \bar{v})$ must cancel at the end.

If we accept the above proposal then the computation of the partition function will involve computing the M5-brane partition function with certain restriction on the 3-form fluxes and picking up (ρ, σ, v) dependence of the result without worrying about any overall divergence or zero coming from bosonic or fermionic zero mode integration. Even then the analysis is not entirely straightforward since the world-volume theory of the wrapped M5-brane resembles that of heterotic string in static gauge Green-Schwarz formulation and most of the explicit genus two computations in heterotic string theory are performed in the Neveu-Schwarz-Ramond formulation. We hope to return to these issues in the near future.

We end by noting that knowing the partition function is important not only for finding the explicit form of the dyon spectrum but also for understanding why as an analytic function the partition function retains its form as we cross a wall of marginal stability.

Acknowledgments

We would like to thank Miranda Cheng and Atish Dabholkar for useful discussions. We would also like to thank Miranda Cheng for her critical comments on an earlier version of

JHEP03(2009)151

the manuscript which helped us fix a sign error. The research of A.S. was supported in part by the JC Bose fellowship. We would also like to acknowledge the hospitality of the 'Monsoon workshop on string theory' at TIFR where part of the work was done.

References

[1] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026] [SPIRES].

[2] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287] [SPIRES].

[3] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094] [SPIRES].

[4] D. Gaiotto, *Re-recounting dyons in $N = 4$ string theory*, hep-th/0506249 [SPIRES].

[5] D. Shih and X. Yin, *Exact black hole degeneracies and the topological string*, *JHEP* **04** (2006) 034 [hep-th/0508174] [SPIRES].

[6] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147] [SPIRES].

[7] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, *JHEP* **06** (2006) 064 [hep-th/0602254] [SPIRES].

[8] A. Dabholkar and S. Nampuri, *Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift*, *JHEP* **11** (2007) 077 [hep-th/0603066] [SPIRES].

[9] J.R. David and A. Sen, *CHL dyons and statistical entropy function from $D1 - D5$ system*, *JHEP* **11** (2006) 072 [hep-th/0605210] [SPIRES].

[10] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in $N = 4$ supersymmetric type-II string theories*, *JHEP* **11** (2006) 073 [hep-th/0607155] [SPIRES].

[11] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in generic $N = 4$ supersymmetric $Z(N)$ orbifolds*, *JHEP* **01** (2007) 016 [hep-th/0609109] [SPIRES].

[12] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, *JHEP* **12** (2007) 087 [hep-th/0612011] [SPIRES].

[13] A. Sen, *Walls of marginal stability and dyon spectrum in $N = 4$ supersymmetric string theories*, *JHEP* **05** (2007) 039 [hep-th/0702141] [SPIRES].

[14] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023 [hep-th/0702150] [SPIRES].

[15] N. Banerjee, D.P. Jatkar and A. Sen, *Adding charges to $N = 4$ dyons*, *JHEP* **07** (2007) 024 [arXiv:0705.1433] [SPIRES].

[16] A. Sen, *Two centered black holes and $N = 4$ dyon spectrum*, *JHEP* **09** (2007) 045 [arXiv:0705.3874] [SPIRES].

[17] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, *JHEP* **09** (2007) 070 [arXiv:0706.2363] [SPIRES].

[18] A. Sen, *Rare decay modes of quarter BPS dyons*, *JHEP* **10** (2007) 059 [arXiv:0707.1563] [SPIRES].

JHEP03(2009)151

[19] A. Mukherjee, S. Mukhi and R. Nigam, *Dyon death eaters*, *JHEP* **10** (2007) 037 [arXiv:0707.3035] [SPIRES].

[20] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [arXiv:0708.1270] [SPIRES].

[21] A. Mukherjee, S. Mukhi and R. Nigam, *Kinematical analogy for marginal dyon decay*, arXiv:0710.4533 [SPIRES].

[22] S. Banerjee and A. Sen, *Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T^6*, *JHEP* **03** (2008) 022 [arXiv:0712.0043] [SPIRES].

[23] S. Banerjee and A. Sen, *S-duality action on discrete T-duality invariants*, *JHEP* **04** (2008) 012 [arXiv:0801.0149] [SPIRES].

[24] S. Banerjee, A. Sen and Y.K. Srivastava, *Generalities of quarter BPS dyon partition function and dyons of torsion two*, *JHEP* **05** (2008) 101 [arXiv:0802.0544] [SPIRES].

[25] S. Banerjee, A. Sen and Y.K. Srivastava, *Partition functions of torsion > 1 dyons in heterotic string theory on T^6*, *JHEP* **05** (2008) 098 [arXiv:0802.1556] [SPIRES].

[26] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, arXiv:0803.2692 [SPIRES].

[27] A. Sen, *Wall crossing formula for $N = 4$ dyons: a macroscopic derivation*, *JHEP* **07** (2008) 078 [arXiv:0803.3857] [SPIRES].

[28] M.C.N. Cheng and E.P. Verlinde, *Wall crossing, discrete attractor flow and Borcherds algebra*, arXiv:0806.2337 [SPIRES].

[29] O. Aharony and A. Hanany, *Branes, superpotentials and superconformal fixed points*, *Nucl. Phys.* **B 504** (1997) 239 [hep-th/9704170] [SPIRES].

[30] O. Aharony, A. Hanany and B. Kol, *Webs of (p, q) 5-branes, five dimensional field theories and grid diagrams*, *JHEP* **01** (1998) 002 [hep-th/9710116] [SPIRES].

[31] A. Sen, *String network*, *JHEP* **03** (1998) 005 [hep-th/9711130] [SPIRES].

[32] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049] [SPIRES].

[33] F. Denef, *On the correspondence between D-branes and stationary supergravity solutions of type-II Calabi-Yau compactifications*, hep-th/0010222 [SPIRES].

[34] F. Denef, B.R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, *JHEP* **05** (2001) 012 [hep-th/0101135] [SPIRES].

[35] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [hep-th/0206072] [SPIRES].

[36] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, hep-th/0304094 [SPIRES].

[37] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, hep-th/0702146 [SPIRES].

[38] A. Gregori et al., *R^2 corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys.* **B 510** (1998) 423 [hep-th/9708062] [SPIRES].

[39] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, U.S.A. (1978).

JHEP03(2009)151